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ASYMPTOTIC MODELS FOR THE DISCRETE OPTIMAL CONTROL OF THE DEFORMATION OF AN ELASTIC MEMBRANE

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This paper considers the singularly perturbed static problem of the optimal control of the deformation of an elastic membrane by means of external loads (control without constraints) applied to several small areas distant from each other. The objective functional is equal to the sum of the square of the root-mean-square approximation error and the square of the norm of the external load. Asymptotic models are constructed using the method of matched asymptotic expansions.

Key words: *elastic membrane, control of quasipoint loads, asymptotic models.*

INTRODUCTION

As is known, in contrast to an elastic plate (in the Kirchhoff) theory), an elastic membrane does not take up a concentrated load, i.e., in the vicinity of the point of application of a concentrated force, the membrane deflection function has a logarithmic singularity and the displacement at this point is theoretically infinite. However, if an external load is distributed over a small area, it is possible to speak of quasipoint loads and consider the corresponding approximate mathematical models. In the present paper, we construct asymptotic models for the optimal control of the deformation of an elastic membrane by means of quasipoint loads.

Khludnev [1] studied the problem of controlling external loads for a shallow shell with a crack in the case where the objective functional characterizes the crack opening. A number of optimal control problems for elastic plates were studied in [2]. Sokołowski and Żochowski [3] investigated the optimal control problem without constraints for an elastic membrane in the case where the objective functional is the sum of the square of the root-mean-square approximation error and the square of the norm of the controlling external loads. Nazarov [4] performed an asymptotic analysis of the deformation of an elastic membrane above a system of several small cylindrical supports for the case where the membrane is acted upon by the singular responses of the supports of arbitrary cross sections that are concentrated at the sharp edges of the supports.

In the present paper, a formal asymptotic representation of the solution of the optimal control problem [3] is constructed for the case where the controlling external loads are applied at several small areas distant from each other and from the membrane contour. The discrete optimal control problem is considered under the assumption that the controlling loads are distributed uniformly. In this case, the shape of the objective functional is different from that in [3]. The problem of the optimal control of the deformation of an elastic membrane by means of several spherical dies is investigated. In this problem, the dimensions and arrangement of the small areas over which the load is transferred from the dies to the membrane are not known in advance.

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1. CONTROL OF QUASIPOINT LOADS

1.1. Formulation of the Problem. Let Ω be a bounded domain with a piecewise smooth boundary Γ on a plane $\mathbb{R}^2$. Inside the domain Ω, we choose points $P^1, \dots, P^N$ with coordinates (x_1^j, x_2^j), $j = 1, \dots, N$. The least distance between the points P^j and P^k for $k \neq j$ will be denoted by d. We assume that these points are separated from the contour Γ by distances not smaller than d. In addition, let ω^j be a simply connected domain on a plane that contains the coordinate origin and is enclosed in a circle of diameter d. We introduce a small positive parameter ε and set

$$\omega_\varepsilon^j = \Big\{x = (x_1, x_2)\colon \varepsilon^{-1}(x - x^j) \in \omega^j\Big\}.$$

Let $\chi_\varepsilon^j(x)$ designate the characteristic function of the domain ω_ε^j, i.e., $\chi_\varepsilon^j(x) = 1$ if $x \in \omega_\varepsilon^j$ and $\chi_\varepsilon^j(x) = 0$ if $x \notin \omega_\varepsilon^j$.

We assume that an elastic membrane with a uniform tension T occupies the domain Ω and is clamped on the contour Γ. We consider the problem of its deformation under the action of a uniform surface pressure with densities $q_1, \dots, q_N$ which is distributed over small areas $\omega_\varepsilon^1, \dots, \omega_\varepsilon^N$:

$$-T\Delta_x u(x) = \sum_{j=1}^{N} \chi_\varepsilon^j(x) q_j, \qquad x \in \Omega; \tag{1.1}$$

$$u(x) = 0, \qquad x \in \Gamma. \tag{1.2}$$

Resultant pressure q_j applied to the area ω_ε^j will be denoted by Q_j:

$$Q_j = \iint_{\omega_\varepsilon^j} q_j \, dx_1 \, dx_2. \tag{1.3}$$

In the case of the uniform load, we have

$$Q_j = q_j |\omega_\varepsilon^j|. \tag{1.4}$$

We consider the problem of determining the optimal load from the minimum condition for the objective functional

$$I(Q_1, \dots, Q_N) = \frac{1}{2} \sum_{j=1}^{N} \Big(\bar{u}(P^j) - u_0^j\Big)^2 + \alpha_j T^{-2} Q_j^2. \tag{1.5}$$

In Eqs. (1.4) and (1.5), u_0^j and $0 < \alpha_j$ are specified constants ($j = 1, \dots, N$), $|\omega_\varepsilon^j| = \varepsilon^2 |\omega^j|$ is the area of the domain ω_ε^j, and $\bar{u}(P^j)$ is the average displacement on the area ω_ε^j with center at the point P^j:

$$\bar{u}(P^j) = \frac{1}{|\omega_\varepsilon^j|} \iint_{\omega_\varepsilon^j} u(x) \, dx. \tag{1.6}$$

Remark 1. The normalization in the second term of the sum (1.5) is such that the constant α_j (the penalty parameter) is a dimensionless quantity. At the same time, by virtue of relations (1.4) and (1.6), the elastic energy

$$J(u) = \frac{1}{2} \sum_{j=1}^{N} \iint_{\omega_\varepsilon^j} q_j u(x) \, dx$$

stored in the membrane can be written as

$$J(u) = \frac{1}{2} \sum_{j=1}^{N} Q_j \bar{u}(P^j).$$

Thus, the quantities Q_j and $\bar{u}(P^j)$ ($j = 1, \dots, N$) can be treated as the generalized forces and the corresponding generalized displacements.

In other words, the minimization of the functional (1.5) with respect to the quantities $Q_1, \dots, Q_N$ implies the validity of the approximate equalities $\bar{u}(P^j) \approx u_0^j$ ($j = 1, \dots, N$) with the least possible quasipoint loads with densities $|\omega_\varepsilon^j|^{-1} Q_j$ distributed over the areas ω_ε^j ($j = 1, \dots, N$).

1.2. Construction of Asymptotic Representation. We use the method of matched asymptotic expansion (see [4–6]) to construct the main terms of the external (at a distance from the points $P^1, \dots, P^N$) and internal (near the areas $\omega_\varepsilon^1, \dots, \omega_\varepsilon^N$) asymptotic expansions of the solution $u(x)$ of problem (1.1), (1.2) for the case [see formula (1.4)]

$$q_j = |\omega_\varepsilon^j|^{-1} Q_j; \tag{1.7}$$

the quantities Q_j $(j = 1, \dots, N)$ do not depend on the parameter ε.

Passing to the limit as $\varepsilon \to 0$ in relations (1.1) and (1.2), we obtain the first limiting problem

$$\Delta_x v(x) = 0, \quad x \in \Omega \setminus \{P^1, \dots, P^N\}, \qquad v(x) = 0, \quad x \in \Gamma. \tag{1.8}$$

The solution of problem (1.8) is sought in the form

$$v(x) = c_1 G_1(x) + \ldots + c_N G_N(x). \tag{1.9}$$

Here $c_1, \dots, c_N$ are certain constants, $G_j(x)$ is the Green function of the Dirichlet problem that has a pole at the point P^j and satisfies the relations

$$\begin{gathered} \Delta_x G_j(x) = 0, \quad x \in \Omega \setminus P^j, \qquad G_j(x) = 0, \quad x \in \Gamma, \\ G_j(x) = -(2\pi)^{-1} \ln |x - P^j| + O(1), \qquad x \to P^j. \end{gathered} \tag{1.10}$$

Subsequently, we shall need the following asymptotic formula, which specifies the second formula in (1.10):

$$G_j(x) = -(2\pi)^{-1} \ln (|x - P^j|/r_0^j) + O(1), \qquad x \to P^j \tag{1.11}$$

(r_0^j is the internal harmonic radius of the domain Ω with respect to the point P^j).

To construct the internal asymptotic representation $w^j(\xi^j)$, we pass to the "stretched" coordinates in the vicinity of the area ω_ε^j:

$$\xi^j = \varepsilon^{-1}(x - P^j). \tag{1.12}$$

Taking into account the relations $\Delta_x = \varepsilon^{-2}\Delta_\xi$ and $|\omega_\varepsilon^j| = \varepsilon^2 |\omega^j|$ and formula (1.7), from Eq. (1.1) we obtain

$$-T\Delta_\xi w^j(\xi) = \chi^j(\xi) |\omega^j|^{-1} Q_j, \qquad \xi \in \mathbb{R}^2. \tag{1.13}$$

Here $\chi^j(\xi)$ is the characteristic function of the domain ω^j.

For Eq. (1.13), we impose an asymptotic condition at infinity, which is obtained by matching the internal $w^j(\xi^j)$ and external $v(x)$ asymptotic representations based on the asymptotic formula (1.11):

$$w^j(\xi) = -\frac{c_j}{2\pi} \ln \frac{\varepsilon|\xi^j|}{r_0^j} + \sum_{k \neq j} c_k G_k(P^j) + o(1), \qquad |\xi^j| \to \infty. \tag{1.14}$$

With the use of the logarithmic potential with constant density on the area ω^j, the solution of problem (1.13), (1.14) is written as

$$w^j(\xi) = -\frac{T^{-1} Q_j}{2\pi |\omega^j|} \iint\limits_{\omega^j} \ln |\xi - \eta| \, d\eta + \text{const}. \tag{1.15}$$

In expression (1.15), we distinguish the function with zero average on the area ω^j. For this, we introduce the quantity R^j which has the dimension of length and is defined by the formula

$$-\frac{1}{|\omega^j|^2} \iint\limits_{\omega^j} \iint\limits_{\omega^j} \ln |\xi - \eta| \, d\eta \, d\xi = \frac{1}{4} - \ln R^j. \tag{1.16}$$

We set

$$w^j(\xi) = -\frac{T^{-1} Q_j}{2\pi |\omega^j|} \Big(\iint\limits_{\omega^j} \ln |\xi - \eta| \, d\eta + |\omega^j| \Big(\frac{1}{4} - \ln R^j \Big) \Big) + \bar{w}^j(P^j). \tag{1.17}$$

The quantity $\bar{w}^j(P^j)$ has the meaning of the average value of the function $w^j(\xi)$ on the area ω^j:

$$\bar{w}^j(\xi) = \frac{1}{|\omega^j|} \iint\limits_{\omega^j} w^j(\eta)\, d\eta. \tag{1.18}$$

Under the assumption that the origin of the coordinates ξ^j coincides with the center of gravity of the figure ω^j (i.e., the point P^j coincides with the center of gravity of the area ω_ε^j), function (1.17) at infinity behaves as follows:

$$w^j(\xi) = -\frac{Q_j}{2\pi T}\Big(\ln\frac{|\xi|}{R^j} + \frac{1}{4}\Big) + \bar{w}^j(P^j) + O(|\xi|^{-2}), \qquad |\xi| \to \infty. \tag{1.19}$$

From the equality of the terms distinguished in expansions (1.14) and (1.19), we find that $c_j = T^{-1}Q_j$ and obtain the relations

$$\frac{Q_j}{2\pi T}\Big(\ln\frac{r_0^j}{\varepsilon R^j} + \frac{1}{4}\Big) + \sum_{k\neq j} T^{-1}Q_k G_k(P^j) = \bar{w}^j(P^j). \tag{1.20}$$

We introduce the notation

$$G_{jj} = \frac{1}{2\pi}\Big(\ln\frac{r_0^j}{\varepsilon R^j} + \frac{1}{4}\Big), \qquad G_{jk} = G_k(P^j), \quad k \neq j. \tag{1.21}$$

Then, relation (1.20) becomes

$$\bar{w}^j(P^j) = \sum_{k=1}^{N} G_{jk} T^{-1} Q_k. \tag{1.22}$$

It is obvious that the matrix $G = \|G_{jk}\|_{j,k=1}^N$ is symmetric and that for small values of the parameter ε, it is positive definite.

Substitution of the asymptotic representation $\bar{w}^j(P^j)$ of the quantity $\bar{u}(P^j)$ into functional (1.5) yields

$$I^*(Q_1, \ldots, Q_N) = \frac{1}{2}\sum_{k=1}^{N}\Big(\sum_{k=1}^{N} G_{jk}T^{-1}Q_k - u_0^j\Big)^2 + \alpha_j T^{-2} Q_j^2. \tag{1.23}$$

The functional $I^*(Q)$ is an asymptotic representation of the objective functional $I(Q)$.

Remark 2. It is easy to verify that in the case of a circular domain ω^j, the quantity R^j, which is defined by formula (1.16), coincides with the radius of the circle ω^j. In the case of an elliptic area ω^j with semiaxes a^j and b^j, using the calculation results [7, § 15], we obtain

$$R^j = (a^j + b^j)/2. \tag{1.24}$$

From formula (1.24), it follows that in the case of an elliptic domain ω^j, the quantity R^j coincides with its external conformal radius (see, for example, [8, § 1.3]).

1.3. Optimality Condition. Let the vector $Q = (Q_1, \ldots, Q_N)$ and the function $u(Q, x)$ be a solution of the optimal control problem considered, i.e., the vector Q minimizes the functional (1.5), where u is a solution of the boundary-value problem (1.1), (1.2).

We fix the index j and denote by $\delta_j Q$ the partial variation of the control Q:

$$\delta_j Q = (0, \ldots, 0, \delta Q_j, 0, \ldots, 0) \qquad (j = 1, \ldots, N).$$

Then, by virtue of the linearity of problem (1.1), (1.2), the variation of the state of the membrane $\delta_j u = u(Q + \delta_j Q) - u(Q)$ satisfies the following problem:

$$-T\Delta_x \delta_j u(x) = \chi_\varepsilon^j(x)|\omega_\varepsilon^j|^{-1}\delta Q_j, \quad x \in \Omega, \qquad \delta_j u(x) = 0, \quad x \in \Gamma. \tag{1.25}$$

Accordingly, the partial variation of functional (1.5) has the form

$$\delta_j I(Q, \delta_j Q) = \sum_{k=1}^{N}\Big(\bar{u}(P^k) - u_0^k\Big)\delta_j \bar{u}(P^k) + \alpha_j T^{-2} Q_j \delta Q_j, \tag{1.26}$$

where

$$\delta_j \bar{u}(P^k) = \frac{1}{|\omega_\varepsilon^k|}\iint\limits_{\omega_\varepsilon^k} \delta_j u(x)\, dx. \tag{1.27}$$

Let $G(y,x)$ be the Green function of the Dirichlet problem that has a pole at the point $y \in \Omega$ and satisfies problem (1.10) in which the point P^j needs to be replaced by y. Then, the solution of the problem (1.25) can be written as

$$\delta_j u(x) = \frac{\delta Q_j}{T|\omega_\varepsilon^j|} \iint\limits_{\omega_\varepsilon^j} G(y,x)\,dy. \tag{1.28}$$

Substitution of expression (1.28) into relation (1.27) yields

$$\delta_j \bar{u}(P^k) = T^{-1} G_{jk}^\varepsilon\, \delta Q_j, \tag{1.29}$$

where

$$G_{jk}^\varepsilon = |\omega_\varepsilon^k|^{-1} |\omega_\varepsilon^j|^{-1} \iint\limits_{\omega_\varepsilon^k} \iint\limits_{\omega_\varepsilon^j} G(y,x)\,dy\,dx. \tag{1.30}$$

Thus, by virtue of relations (1.26) and (1.29), the particle variation of the objective functional (1.5) becomes

$$\delta_j I(Q, \delta_j Q) = \Big(\sum_{k=1}^N (\bar{u}(P^k) - u_0^k) T^{-1} G_{jk}^\varepsilon + \alpha_j T^{-2} Q_j \Big) \delta Q_j. \tag{1.31}$$

Hence, a necessary condition that the functional (1.5) be stationary on the vector Q [$\delta_j I(Q, \delta_j Q) = 0$ for any partial variation $\delta_j Q$] can be written, in accordance with expression (1.31), as follows:

$$\sum_{k=1}^N (\bar{u}(P^k) - u_0^k) G_{jk}^\varepsilon + \alpha_j T^{-1} Q_j = 0 \qquad (j = 1, \ldots, N). \tag{1.32}$$

We calculate the asymptotics of the quantity G_{jk}^ε as $\varepsilon \to 0$. Let first $k \neq j$. For a fixed point $x \in \omega_\varepsilon^k$ for $y \in \omega_\varepsilon^j$, from the Taylor formula we have

$$G(y,x) = G(P^j,x) + \frac{\partial G}{\partial y_1}(P^j,x)(y_1 - x_1^j) + \frac{\partial G}{\partial y_2}(P^j,x)(y_2 - x_2^j) + O(\varepsilon^2).$$

Because the point P^k is assumed to coincide with the center of gravity of the area ω_ε^k, we obtain

$$G_{jk}^\varepsilon = |\omega_\varepsilon^k|^{-1} \iint\limits_{\omega_\varepsilon^k} G(P^j,x)\,dx + O(\varepsilon^2).$$

Similar reasoning leads to

$$G_{jk}^\varepsilon = G_{jk} + O(\varepsilon^2), \qquad \varepsilon \to 0 \qquad (k \neq j). \tag{1.33}$$

Let now $k = j$. Transforming to the coordinates (1.12), we obtain

$$G_{jj}^\varepsilon = \frac{1}{|\omega^j|^2} \iint\limits_{\omega^j} \iint\limits_{\omega^j} G(P^j + \varepsilon\eta, P^j + \varepsilon\xi)\,d\eta\,d\xi. \tag{1.34}$$

We recall that by the definition of the Green function,

$$G(y,x) = -(2\pi)^{-1} \ln|y - x| + g(y,x),$$

where $g(y,x)$ is a regular function. Therefore,

$$G(P^j + \varepsilon\eta, P^j + \varepsilon\xi) = -(2\pi)^{-1} \ln \varepsilon|\eta - \xi| + g(P^j,P^j) + \varepsilon \sum_{i=1}^2 \frac{\partial g}{\partial y_i}(P^j,P^j)\eta_i + \frac{\partial g}{\partial x_i}(P^j,P^j)\xi_i + O(\varepsilon^2). \tag{1.35}$$

By the definition of the inner harmonic radius, we have

$$g(P^j,P^j) = (2\pi)^{-1} \ln r_0^j. \tag{1.36}$$

Substitution of expansion (1.35) into integral (1.34) with the use of (1.36) yields

$$2\pi G_{jj}^\varepsilon = -\frac{1}{|\omega^j|^2} \iint\limits_{\omega^j} \iint\limits_{\omega^j} \ln|\eta - \xi|\,d\eta\,d\xi + \ln\frac{r_0^j}{\varepsilon} + O(\varepsilon^2), \qquad \varepsilon \to 0.$$

Finally, taking into account Eq. (1.16) and the notation (1.21), we have

$$G^{\varepsilon}_{jj} = G_{jj} + O(\varepsilon^2), \qquad \varepsilon \to 0. \tag{1.37}$$

At the same time, the necessary condition for an extremum of function (1.23) at the point Q is written as $(j = 1, \ldots, N)$

$$\sum_{k=1}^{N} \Big(\sum_{l=1}^{N} G_{lk} T^{-1} Q_l - u_0^k \Big) G_{jk} + \alpha_j T^{-1} Q_j = 0. \tag{1.38}$$

In view of relations (1.22), (1.33), and (1.37), it can be concluded that if the asymptotic relations (1.38) are satisfied, the optimality condition (1.32) is satisfied with accuracy up to $O(\varepsilon^2 |\ln \varepsilon|)$ for $\varepsilon \to 0$.

2. CONTROL OF SMALL DIES

2.1. Formulation of the Problem. Let an elastic membrane Ω under a uniform tension T clamped on the contour Γ be acted upon by a system of N dies in the form of circular paraboloids:

$$\Phi_j(x) = (2R_j)^{-1}[(x_1 - x_1^j)^2 + (x_2 - x_2^j)^2] \qquad (j = 1, \ldots, N). \tag{2.1}$$

Then, the membrane deflection function satisfies the problem (see, for example, [9, 10])

$$\begin{gathered} -T\Delta_x u(x) \ge 0, \qquad u(x) \ge u(P^j) - \Phi_j(x), \\ \Delta_x u(x)[u(x) - u(P^j) + \Phi_j(x)] = 0, \qquad x \in \omega_*^j \quad (j = 1, \ldots, N); \end{gathered} \tag{2.2}$$

$$\Delta_x u(x) = 0, \qquad x \in \Omega \setminus \bigcup_{j=1}^{N} \omega_*^j; \tag{2.3}$$

$$u(x) = 0, \qquad x \in \Gamma. \tag{2.4}$$

Here $u(P^j)$ is the translational displacement of the die with number j, which is to be determined from the specified value of the force acting on the die; ω_*^j is a domain that encompasses the *a priori* unknown contact area ω_ε^j under the die with number j. It can be assumed that ω_*^j coincides with area (2.2) (see [11]).

Problem (2.2)–(2.4) is studied under the assumption of small contact areas. We introduce a small positive parameter ε and set

$$R_j = \varepsilon R_j^* \qquad (j = 1, \ldots, N), \tag{2.5}$$

where the quantities R_j^* are comparable to the characteristic distance d. Then, ω_*^j is a circle of radius $O(\sqrt{\varepsilon}\, d)$.

According to the adopted shape of the die (2.1), the pressure

$$q_j(x) = T\Delta_x \Phi_j(x) \qquad (x \in \omega_\varepsilon^j)$$

transferred by the die to the membrane is uniform over the contact area ω_ε^j:

$$q_j(x) = 2R_j^{-1} T, \qquad x \in \omega_\varepsilon^j. \tag{2.6}$$

In this case, the location of the boundary of the contact area ω_ε^j in the vicinity of the point P^j is not known in advance.

Hence, the force Q_j acting on the die with number j is equal to

$$Q_j = 2R_j^{-1} T |\omega_\varepsilon^j|. \tag{2.7}$$

The quantities Q_j and $u(P^j)$ $(j = 1, \ldots, N)$ can be treated as the generalized forces and their corresponding generalized displacements. In the specification of the forces $Q_1, \ldots, Q_N$, the equilibrium equations (2.7) serve to determine the displacements $u(P^1), \ldots, u(P^N)$.

Let $u_0^1, \dots, u_0^N$ be specified constants. Then, the minimization of the functional

$$I(Q_1, \dots, Q_N) = \frac{1}{2}\sum_{j=1}^{N}(u(P^j) - u_0^j)^2 + \alpha_j T^{-2} Q_j^2 \tag{2.8}$$

with respect to the quantities $Q_1, \dots, Q_N$ implies an approximation to the satisfaction of the equalities $u(P^j) = u_0^j$ $(j = 1, \dots, N)$ with the least possible forces acting on the dies.

2.2. Asymptotic Model. Following [11], where the problem of unilateral contact for one die was investigated in detail, we construct an asymptotic representation of the solutions of problem (2.2)–(2.4), (2.7) for $0 < \varepsilon \ll 1$ using the method of matched expansions.

The external asymptotic representation (at a distance from the points $P^1, \dots, P^N$) can be written as

$$v(x) = \sum_{j=1}^{N} T^{-1} Q_j G_j(x), \tag{2.9}$$

where $G_j(x)$ is the Green function which is a solution of problem (1.10).

In the vicinity of the die with number j, we introduce "stretched" coordinates by the formula

$$\xi^j = \varepsilon^{-1/2}(x - P^j). \tag{2.10}$$

The internal asymptotic representation $w^j(\xi^j)$ satisfies the relations

$$-\Delta_\xi w^j(\xi) \ge 0, \qquad w^j(\xi) \ge u(P^j) - \Phi_j^*(\xi),$$

$$\Delta_\xi w^j(\xi)\Big[w^j(\xi) - u(P^j) + \Phi_j^*(\xi)\Big] = 0, \qquad \xi \in \mathbb{R}^2. \tag{2.11}$$

Here $\Phi_j^*(\xi) = (2R_j^*)^{-1}(\xi_1^2 + \xi_2^2)$. It should be noted that relations (2.11) are obtained from (2.2) and (2.3) with allowance for (2.5) and (2.10).

The matching condition [see also (1.11) and (1.14)] is written as

$$w^j(\xi) = \frac{Q_j}{2\pi T}\ln\frac{r_0^j}{\varepsilon|\xi|} + \sum_{k\ne j} T^{-1} Q_k G_k(P^j) + O(|\xi|^{-1}), \qquad |\xi| \to \infty. \tag{2.12}$$

Problem (2.11), (2.12) admits a solution in closed form. We denote by a_j^* the radius of the contact area in the "stretched" coordinates. Then, under the assumption of continuity of the function $w^j(\xi)$ and its first-order partial derivatives, relations (2.11) are equivalent to the relations

$$w^j(\xi) = u(P^j) - (2R_j^*)^{-1}(a_j^*)^2, \qquad \frac{\partial w^j}{\partial \rho}(\xi) = -(R_j^*)^{-1} a_j^*, \qquad \rho \equiv |\xi| = a_j^*. \tag{2.13}$$

In this case, $\Delta w^j(\xi) = 0$ for $|\xi| > a_j^*$; based on (2.12), we have

$$w^j(\xi) = \frac{Q_j}{2\pi T}\ln\frac{r_0^j}{\varepsilon|\xi|} + \sum_{k\ne j} T^{-1} Q_k G_k(P^j), \qquad |\xi| \ge a_j^*. \tag{2.14}$$

Satisfying conditions (2.13), we obtain the dependences

$$(a_j^*)^2 = (2\pi T)^{-1} Q_j R_j^*; \tag{2.15}$$

$$u(P^j) = \frac{Q_j}{4\pi T} + \frac{Q_j}{2\pi T}\ln\frac{r_0^j}{\varepsilon a_j^*} + \sum_{k\ne j} T^{-1} Q_k G_k(P^j). \tag{2.16}$$

Eliminating the parameter a_j^* from Eq. (2.16) with the use of (2.15) and taking into account (2.5), we have the equation

$$\frac{Q_j}{4\pi T}\Big(1 + \ln\frac{2\pi T (r_0^j)^2}{Q_j R_j}\Big) + \sum_{k\ne j} T^{-1} Q_k G_k(P^j) = u(P^j). \tag{2.17}$$

Thus, in the asymptotic model (2.17), which approximately describes the pressure of the system of small spherical dies on an elastic membrane, the necessary condition for an extremum of function (2.8) has the following form ($j = 1, \ldots, N$):

$$\Big[\frac{Q_j}{4\pi T}\Big(1 + \ln\frac{2\pi T (r_0^j)^2}{Q_j R_j}\Big) + \sum_{k\neq j} T^{-1} Q_k G_k(P^j) - u_0^j\Big]\frac{1}{4\pi}\,\ln\frac{2\pi T (r_0^j)^2}{Q_j R_j}$$

$$+ \sum_{k\neq j}\Big[\frac{Q_k}{4\pi T}\Big(1 + \ln\frac{2\pi T (r_0^k)^2}{Q_k R_k}\Big) + \sum_{l\neq k} T^{-1} Q_l G_l(P^k)\Big] G_j(P^k) + \alpha_j T^{-1} Q_j = 0. \tag{2.18}$$

The system of N nonlinear equations (2.18) serves to seek the optimal controlling forces $Q_1, \ldots, Q_N$ from the specified displacements $u_0^1, \ldots, u_0^N$.

Remark 3. Equations (2.17) and (2.18) remain valid in the case of dies in the form of elliptic paraboloids if the quantity R_j is replaced by the arithmetic average of the curvature radii of the main normal sections of the die surface at its vertex (see [11]).

3. ASYMPTOTIC SOLUTION OF THE DISCRETE OPTIMAL CONTROL PROBLEM

3.1. Formulation of the Problem and Optimality Conditions. We assume that an elastic membrane Ω under a uniform tension T is clamped on the contour Γ, and on small areas $\omega_\varepsilon^1, \ldots, \omega_\varepsilon^N$, it is acted upon by a surface load $q_1(x), \ldots, q_N(x)$. Using the notation introduced in Sec. 1.1, the problem of determining the membrane deflection $u(x)$ is written as

$$-T\Delta_x u(x) = \sum_{j=1}^{N} \chi_\varepsilon^j(x) q_j(x), \qquad x \in \Omega; \tag{3.1}$$

$$u(x) = 0, \qquad x \in \Gamma. \tag{3.2}$$

In addition, let $u_0^1, \ldots, u_0^N$ be specified constants. We consider the problem of determining the control $q_1(x), \ldots, q_N(x)$ such that the solution $u(x)$ of problem (3.1), (3.2) differs insignificantly from the constants $u_0^1, \ldots, u_0^N$ on the small areas $\omega_\varepsilon^1, \ldots, \omega_\varepsilon^N$, respectively. We seek to find the least control by minimizing the objective functional [compare with (1.5) and (1.4)]

$$I(q_1, \ldots, q_N) = \frac{1}{2}\sum_{j=1}^{N} \iint\limits_{\omega_\varepsilon^j} \Big[(u(x) - u_0)^2 + \alpha_j T^{-2} |\omega_\varepsilon^j|^2 q_j(x)^2\Big]\, dx. \tag{3.3}$$

The optimality conditions for the objective functional of more general form than (3.3) were obtained in [3]. In the case considered, the optimal control problem (3.1)–(3.3) reduces to the following system of coupled differential equations:

$$-\Delta_x u(x) = -\sum_{j=1}^{N} \chi_\varepsilon^j(x) \alpha_j^{-1} T |\omega_\varepsilon^j|^{-2} p(x), \qquad x \in \Omega; \tag{3.4}$$

$$-\Delta_x p(x) = \sum_{j=1}^{N} \chi_\varepsilon^j(x) T^{-1}(u(x) - u_0^j), \qquad x \in \Omega; \tag{3.5}$$

$$u(x) = 0, \quad x \in \Gamma, \qquad p(x) = 0, \quad x \in \Gamma. \tag{3.6}$$

Here $p(x)$ is a conjugate function. In this case, the control is determined by the solution of problem (3.4)–(3.6) according to the dependence

$$q_j(x) = -\frac{T^2}{\alpha_j |\omega_\varepsilon^j|^2}\, p(x), \qquad x \in \omega_\varepsilon^j \quad (j = 1, \ldots, N). \tag{3.7}$$

We note that Eqs. (3.4) and (3.5) were derived under the assumption of continuity of the functions $u(x)$ and $p(x)$ and their first-order partial derivatives.

We examine the behavior of the solution of problem (3.4)–(3.6) for $\varepsilon \to 0$ using the method of matched expansions.

3.2. Construction of the Asymptotics. Passing to the limit as $\varepsilon \to 0$ in (3.4)–(3.6), we obtain the first limiting problem

$$\Delta_x v(x) = 0, \quad x \in \Omega \setminus \{P^1, \dots, P^N\}, \qquad v(x) = 0, \quad x \in \Gamma; \tag{3.8}$$

$$\Delta_x p_0(x) = 0, \quad x \in \Omega \setminus \{P^1, \dots, P^N\}, \qquad p_0(x) = 0, \quad x \in \Gamma. \tag{3.9}$$

From (3.8) and (3.9), it follows that outside the areas $\omega_\varepsilon^1, \dots, \omega_\varepsilon^N$, Eqs. (3.4) and (3.5) coincide with the Laplace equation, which is invariant under coordinate stretching. Therefore, we fix the index j and consider Eqs. (3.4) and (3.5) on the area ω_ε^j.

In the transition to the "stretched" coordinates

$$\xi^j = \varepsilon^{-1}(x - P^j), \tag{3.10}$$

Eqs. (3.4) and (3.5) are transformed as follows:

$$\varepsilon^{-2}\Delta_\xi u = \alpha_j^{-1} T \varepsilon^{-4} |\omega^j|^{-2} p, \qquad \xi \in \omega^j; \tag{3.11}$$

$$-\varepsilon^{-2}\Delta_\xi p = T^{-1}(u - u_0^j), \qquad \xi \in \omega^j. \tag{3.12}$$

Here the argument $x = P^j + \varepsilon\xi^j$ of the functions u and p is omitted to simplify the presentation.

Because $|\omega_\varepsilon^j| = \varepsilon^2 |\omega^j|$, Eq. (3.7) on the area ω^j can be represented as

$$q_j = -\frac{T^2}{\alpha_j |\omega^j|^2}\,\varepsilon^{-4} p, \qquad \xi \in \omega^j. \tag{3.13}$$

Accordingly, Eq. (3.11) becomes

$$-\varepsilon^{-2} T \Delta_\xi u = q_j, \qquad \xi \in \omega^j. \tag{3.14}$$

We set

$$q_j = \varepsilon^{-2} q_j^*(\xi), \qquad \xi \in \omega^j. \tag{3.15}$$

In this case, the total load on the area ω^j is equal to [see (1.3)]

$$Q_j = \iint\limits_{\omega^j} q_j^*(\xi)\, d\xi. \tag{3.16}$$

In view of relations (3.14) and (3.15), the internal asymptotic representation for the function $u(x)$ in the vicinity of the area ω_ε^j is represented as the logarithmic potential

$$w^j(\xi) = -\frac{1}{2\pi} \iint\limits_{\omega^j} T^{-1} q_j^*(\eta) \ln|\xi - \eta|\, d\eta + c_j, \tag{3.17}$$

where c_j is a constant.

For function (3.17), the following asymptotic formula is valid:

$$w^j(\xi) = -\frac{Q_j}{2\pi T} \ln|\xi| + c_j + O(|\xi|^{-1}), \qquad |\xi| \to \infty. \tag{3.18}$$

From the condition of matching of the internal asymptotic representation $w^j(\xi)$ and external asymptotic representation $v(x)$, using formula (3.18), we obtain the following representation for $v(x)$:

$$v(x) = \sum_{j=1}^{N} T^{-1} Q_j G_j(x). \tag{3.19}$$

Here $G_j(x)$ is the Green function that satisfies problem (1.10). At the same time, for function (3.19) with $x \to P^j$, the following expansion holds [see (1.11)]:

$$v(x) = -\frac{Q_j}{2\pi T} \ln \frac{|x - P^j|}{r_0^j} + \sum_{k \neq j} T^{-1} Q_k G_k(P^j) + O(|x - P^j|). \tag{3.20}$$

Transforming to coordinates (3.10) in relation (3.20) and comparing the result with expansion (3.18), we determine the constant c_j:

$$c_j = -\frac{Q_j}{2\pi T} \ln \frac{\varepsilon}{r_0^j} + \sum_{k \neq j} T^{-1} Q_k G_k(P^j). \tag{3.21}$$

Let us determine the function $q_j^*(\xi)$. In view of relations (3.13) and (3.15), Eq. (3.12) is written as

$$\alpha_j T^{-1} |\omega^j|^2 \Delta_\xi q_j^*(\xi) = w^j(\xi) - u_0^j, \qquad \xi \in \omega^j. \tag{3.22}$$

It should be noted that in the notation of Eq. (3.22), the function u is replaced by its internal asymptotic representation. Similarly, Eq. (3.14) becomes

$$-T \Delta_\xi w^j(\xi) = q_j^*(\xi), \qquad \xi \in \omega^j. \tag{3.23}$$

The solution of Eq. (3.22), in turn, is written as the logarithmic potential

$$q_j^*(\xi) = \frac{1}{2\pi} \iint\limits_{\omega^j} \alpha_j^{-1} T |\omega^j|^{-2} (w^j(\eta) - u_0^j) \ln |\xi - \eta| \, d\eta + c_j^*, \tag{3.24}$$

where c_j^* is a constant.

According to formulas (3.24), (3.13), and (3.15), we obtain

$$p_0^j(\xi) = -\frac{\varepsilon^2}{2\pi T} \iint\limits_{\omega^j} (w^j(\eta) - u_0^j) \ln |\xi - \eta| \, d\eta - \varepsilon^2 \alpha_j |\omega^j|^2 T^{-2} c_j^*. \tag{3.25}$$

By the construction, function (3.25) is harmonic in the domain of $\mathbb{R}^2 \setminus \overline{\omega}^j$ and can serve as an internal asymptotic representation of the function p in the vicinity of the area ω_ε^j.

We determine the constant c_j^* by matching the function $p_0^j(\xi)$ to the function $p_0(x)$ — the external asymptotic representation of the function p. By virtue of the asymptotic formula

$$p_0^j(\xi) = -\frac{\varepsilon^2}{2\pi T} \ln |\xi| \iint\limits_{\omega^j} (w^j(\eta) - u_0^j) \, d\eta - \varepsilon^2 \alpha_j |\omega^j|^2 T^{-2} c_j^* + O(|\xi|^{-1}),$$

we have

$$p_0(x) = \varepsilon^2 T^{-1} |\omega^j| \sum_{j=1}^{N} (\bar{w}^j - u_0^j) G_j(x).$$

Here

$$\bar{w}^j = \frac{1}{|\omega^j|} \iint\limits_{\omega^j} w^j(\eta) \, d\eta. \tag{3.26}$$

Using the expansion

$$p_0(x) = -\frac{\varepsilon^2 |\omega^j|}{2\pi T} (\bar{w}^j - u_0^j) \ln \frac{|x - P^j|}{r_0^j} + \frac{\varepsilon^2}{T} \sum_{k \neq j} (\bar{w}^k - u_0^k) G_k(P^j) + O(|x - P^j|),$$

which is valid for $x \to P^j$, we find

$$c_j^* = \frac{T}{\alpha_j |\omega^j|^2} \Big(\frac{|\omega^j|}{2\pi} (\bar{w}^j - u_0^j) \ln \frac{\varepsilon}{r_0^j} - \sum_{k \neq j} (\bar{w}^k - u_0^k) G_k(P^j) \Big). \tag{3.27}$$

Finally, taking into account relations (3.21) and (3.27), we have

$$
\begin{gathered}
Tw^j(\xi) = -\frac{1}{2\pi} \iint\limits_{\omega^j} q_j^*(\eta) \ln \frac{\varepsilon|\xi-\eta|}{r_0^j} \, d\eta + \sum_{k\neq j} G_{jk} \iint\limits_{\omega^k} q_k^*(\eta)\, d\eta, \\
\frac{\alpha_j |\omega^j|^2}{T} q_j^*(\xi) = \frac{1}{2\pi} \iint\limits_{\omega^j} (w^j(\eta) - u_0^j) \ln \frac{\varepsilon|\xi-\eta|}{r_0^j} \, d\eta - \sum_{k\neq j} G_{jk} \Big(\frac{1}{|\omega^k|} \iint\limits_{\omega^k} w^k(\eta)\, d\eta - u_0^k \Big).
\end{gathered}
\tag{3.28}
$$

Thus, the asymptotic representation of the solution of problem (3.4)–(3.6) constructed above contains the functions $q_j^*(\xi)$ and $w^j(\xi)$ specified on the area ω^j $(j = 1, \dots, N)$ for which the system of coupled integral equations (3.28) is obtained.

3.3. Asymptotic Model for the Case of Circular Control Areas. Let ω^j be a circle of radius a^j with center at the coordinate origin on the plane of the "stretched" coordinates. Then, according to formulas (3.17), (3.21) and (3.25), (3.27) for $\rho = |\xi| \geq a_j$, the following representations are valid:

$$
w^j(\xi) = -\frac{Q_j}{2\pi T} \ln \frac{\varepsilon \rho}{r_0^j} + \sum_{k\neq j} T^{-1} Q_k G_{jk}; \tag{3.29}
$$

$$
p_0^j(\xi) = -\frac{\varepsilon^2}{2\pi T} |\omega^j| (\bar{w}^j - u_0^j) \ln \frac{\varepsilon\rho}{r_0^j} + \frac{\varepsilon^2}{T} \sum_{k\neq j} (\bar{w}^k - u_0^k) G_{jk}. \tag{3.30}
$$

At the same time, for $0 \leq \rho < a_j$, we have Eqs. (3.22) and (3.23), which imply the relations

$$
\begin{gathered}
\Delta_\xi \Delta_\xi (w^j(\xi) - u_0^j) + \lambda_j^4 (w^j(\xi) - u_0^j) = 0, \qquad \xi \in \omega^j, \\
\Delta_\xi \Delta_\xi q_j^*(\xi) + \lambda_j^4 q_j^*(\xi) = 0, \qquad \xi \in \omega^j.
\end{gathered}
\tag{3.31}
$$

Here $\lambda_j^4 = \alpha_j^{-1} |\omega^j|^{-2}$.

Under conditions of circular symmetry, the solutions of Eqs. (3.31) are expressed in terms of Kelvin functions (see, for example, [12, Part 2, Chapter 1, § 2])

$$
w^j(\xi) - u_0^j = A_j \operatorname{ber}(\lambda_j \rho) + B_j \operatorname{bei}(\lambda_j \rho); \tag{3.32}
$$

$$
q_j^*(\xi) = A_j^* \operatorname{ber}(\lambda_j \rho) + B_j^* \operatorname{bei}(\lambda_j \rho). \tag{3.33}
$$

Here $\operatorname{ber}(x)$ and $\operatorname{bei}(x)$ are Kelvin functions (of zero order) defined by the expansions (see formula (8.564) in [13])

$$
\operatorname{ber}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{2^{4k}[(2k)!]^2}, \qquad \operatorname{bei}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{2^{4k+2}[(2k+1)!]^2}.
$$

By virtue of relations (3.13) and (3.15), the following equality holds:

$$
p_0^j(\xi) = -\varepsilon^2 \alpha_j T^{-2} |\omega^j|^2 q_j^*(\xi), \qquad \xi \in \omega^j. \tag{3.34}
$$

According to representations (3.29) and (3.32), the continuity condition for the function $w^j(\xi)$ and its derivative on the circle $\rho = a_j$ is written as

$$
\begin{gathered}
A_j \operatorname{ber}'(\lambda_j a_j) + B_j \operatorname{bei}'(\lambda_j a_j) = -(2\pi T \lambda_j a_j)^{-1} Q_j, \\
A_j \operatorname{ber}(\lambda_j a_j) + B_j \operatorname{bei}(\lambda_j a_j) = -u_0^j - \frac{Q_j}{2\pi T} \ln \frac{\varepsilon a_j}{r_0^j} + \sum_{k\neq j} T^{-1} Q_k G_{jk}.
\end{gathered}
\tag{3.35}
$$

Similarly, according to representations (3.30), (3.34), and (3.33), the continuity condition for the function $p_0^j(\xi)$ and its derivative for $\rho = a_j$ is written as

$$
\begin{gathered}
A_j^* \operatorname{ber}'(\lambda_j a_j) + B_j^* \operatorname{bei}'(\lambda_j a_j) = \frac{T(\bar{w}^j - u_0^j)}{2\pi \alpha_j |\omega^j| \lambda_j a_j}, \\
A_j^* \operatorname{ber}(\lambda_j a_j) + B_j^* \operatorname{bei}(\lambda_j a_j) = \frac{T}{\alpha_j |\omega^j|^2} \Big((\bar{w}^j - u_0^j) \frac{|\omega^j|}{2\pi} \ln \frac{\varepsilon a_j}{r_0^j} - \sum_{k\neq j} (\bar{w}^k - u_0^k) G_{jk} \Big).
\end{gathered}
\tag{3.36}
$$

Substitution of expressions (3.33) into formula (3.16) yields

$$Q_j = 2\pi \int_0^{a_j} \Big(A_j^* \operatorname{ber}(\lambda_j \rho) + B_j^* \operatorname{bei}(\lambda_j \rho) \Big) \rho \, d\rho. \tag{3.37}$$

Using the formulas (see [14, Chapter 3, § 6])

$$\int_0^x \xi \operatorname{ber}(\xi)\, d\xi = x \operatorname{bei}'(x), \qquad \int_0^x \xi \operatorname{bei}(\xi)\, d\xi = -x \operatorname{ber}'(x),$$

from relation (3.37), we have

$$Q_j = 2\pi \lambda_j^{-1} a_j \Big(A_j^* \operatorname{bei}'(\lambda_j a_j) - B_j^* \operatorname{ber}'(\lambda_j a_j) \Big). \tag{3.38}$$

Similarly, substitution of expression (3.32) into formula (3.26) yields

$$\bar{w}^j - u_0^j = 2(\lambda_j a_j)^{-1} \Big(A_j \operatorname{bei}'(\lambda_j a_j) - B_j \operatorname{ber}'(\lambda_j a_j) \Big). \tag{3.39}$$

Thus, the substitution of expressions (3.38) and (3.39) into Eqs. (3.35) and (3.36) reduces the problem of determining the optimal control to a system of $4N$ linear algebraic equations for the coefficients A_j, B_j and A_j^*, B_j^* $(j = 1, \ldots, N)$. In turn, determining the coefficients A_j and B_j from the system of two equations (3.35) and the coefficients A_j^* and B_j^* from system (3.36) and substituting the result into Eqs. (3.38) and (3.39), we obtain a system of $2N$ equations for the quantities Q_j and $\bar{w}^j$ $(j = 1, \ldots, N)$, which have a mechanical meaning.

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